* Normal shock wave:

→ It is a Compression front in supersonic flow-field

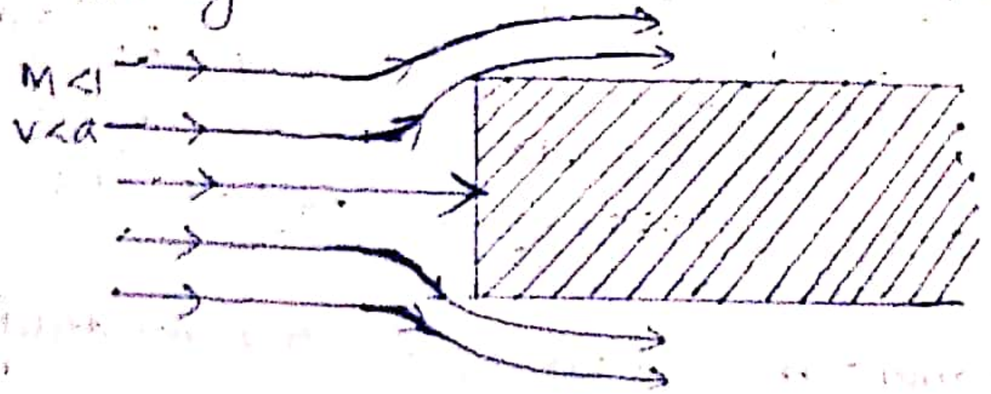
→ Due to the formation of this it will bring abrupt changes in fluid properties.

Now we are considering the flow over a cylinder.

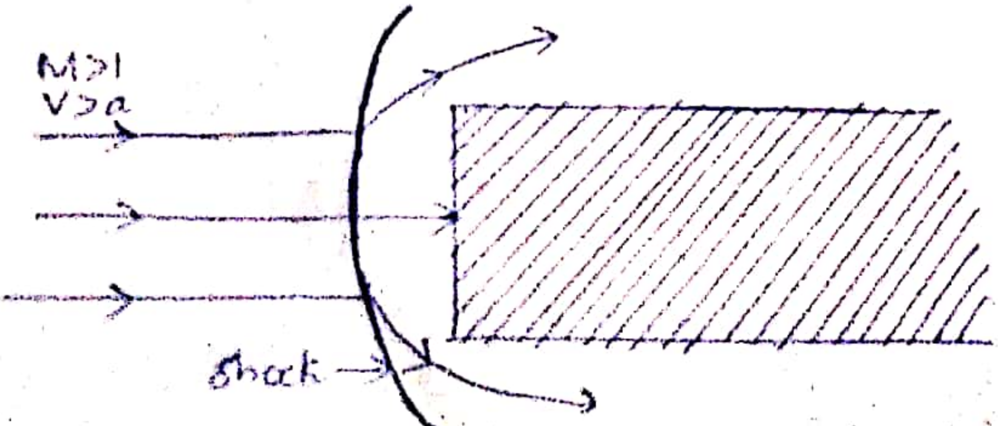
eg.

*Case I: flow is subsonic ($M<1$) which means $V<a$.

In this case, signal carrying wave (a) travels faster than fluid flow. It will when the molecules are moving gives signals to molecules. It (molecules) will orient themselves and move around the cylinder.



(a) Subsonic flow



(b) Supersonic flow

Streamlines around a blunt faced cylinder in Subsonic & Supersonic flows.

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*Case II : When the flow is supersonic ($M>1$) i.e., $V>a$.

Before getting the signal, the molecules will get reflected. And after a short distance, it will get rejoined, which forms a compression front which is called shock wave. The flow behind the normal shock wave is subsonic flow.

Flow velocity is reduced due to presence of obstruction i.e. shock wave.

The formation of shock takes place only when the flow gets impinged and rebounded.

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* Flow through a Normal shock:

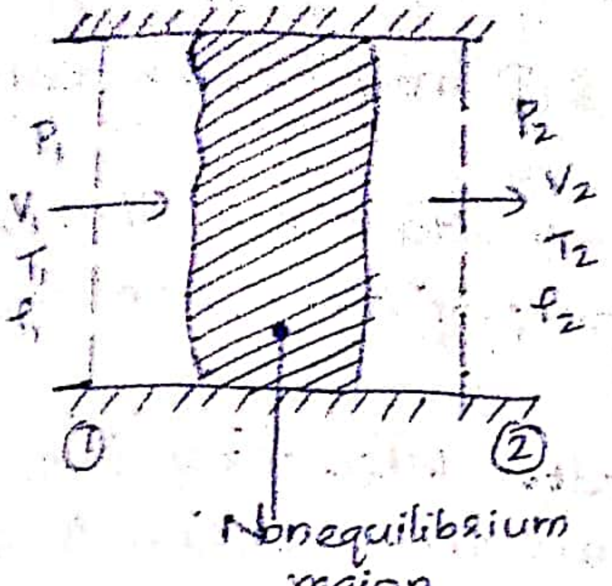

fig: (a)

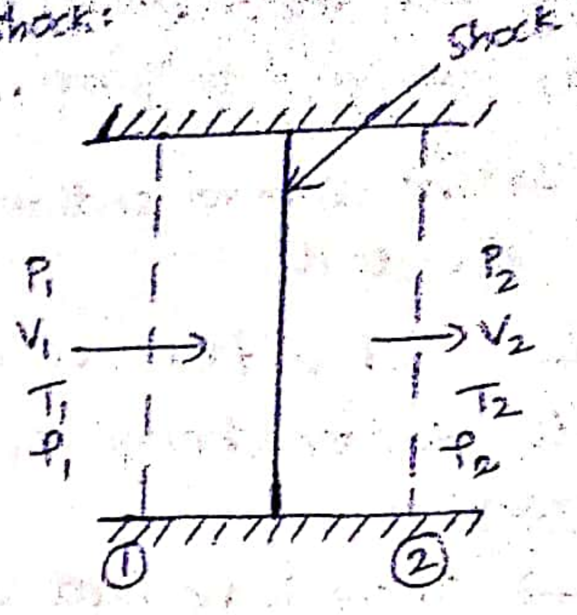

fig: (b).

• Equations of motion for a normal shock wave: (Prandtl's relation).

For a quantitative analysis of changes across a normal shock wave, let us consider an adiabatic, constant flow through a nonequilibrium region (fig a)

Let section ① & ② away from the non equilibrium region

By continuity Equation:

$$\boxed{\rho_1 V_1 = \rho_2 \cdot V_2} \longrightarrow ①$$ ($\because$ Area is constant ie., $A_1 = A_2$).

The momentum equation:

$$\boxed{P_1 + \rho_1 V_1^2 = P_2 + \rho_2 V_2^2} \longrightarrow ②$$

The energy equation:

$$\boxed{h_1 + \frac{1}{2} V_1^2 = h_2 + \frac{1}{2} V_2^2} \longrightarrow ③$$ ($\because$ adiabatic & constant flow).

There is no restriction on the size OR the details of the nonequilibrium region, it may be idealized by vanishin-

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gly thin region (fig (b)) across which the flow paramet-ers are said to 'jump'.

→ The control section ① & ② may also be brought close to it.

→ Such a front or discontinuity across which there is sudden change in flow properties is called shock wave.

→ There is no heat added or taken away from the flow, hence the flow across the shock wave is adia-batic.

⇒ Normal shock relation for a perfect gas:

The equation of state is given by:

$$\boxed{P = \rho R T} \longrightarrow ④$$

and enthalpy, $\boxed{h = C_p T} \longrightarrow ⑤$

5 set of Eqns with 5 unknowns: P_2, ρ_2, T_2, V_2 and h_2. (since we know the inlet parameters).

Equations ① & ③ are the general equations for a normal shock wave. And for a perfect gas it is possible to obtain explicit solution in terms of mach number 'M_1' ahead of the shock.

Divide Eqn ② by Eqn ①:

$$\frac{P_1}{\rho_1 V_1} + \frac{\rho_1 V_1^2}{\rho_1 V_1} = \frac{P_2}{\rho_2 V_2} + \frac{\rho_2 V_2^2}{\rho_2 V_2}.$$

$$\frac{P_1}{\rho_1 V_1} + V_1 = \frac{P_2}{\rho_2 V_2} + V_2$$

$$\boxed{\frac{P_1}{\rho_1 V_1} - \frac{P_2}{\rho_2 V_2} = V_2 - V_1} \longrightarrow ⑥.$$

In the case of adiabatic process, speed of sound,

$$a = \sqrt{\gamma \frac{P}{\rho}} \Longrightarrow a^2 = \frac{\gamma P}{\rho} \Rightarrow \frac{a^2}{\gamma} = \frac{P}{\rho}.$$

Sub in eqn ⑥:

$$\boxed{\frac{a_1^2}{\gamma V_1} - \frac{a_2^2}{\gamma V_2} = V_2 - V_1.} \longrightarrow ⑦$$

{ 'a' will be different in different mediums.

Consider the energy equation ③:

$$h_1 + \frac{1}{2} V_1^2 = h_2 + \frac{1}{2} V_2^2.$$

where $h = C_p T$.

where $\boxed{C_p = \frac{R\gamma}{\gamma - 1} \quad ; \quad P = \rho R T. \quad T = \frac{P}{\rho R}}$

Sub in 'h':

$$h = \frac{R\gamma}{\gamma-1} \cdot \frac{P}{\rho R}$$

$$= \frac{R\gamma}{\gamma-1} \times \frac{a^2}{\gamma R}.$$

where $a = \sqrt{\frac{\gamma P}{\rho}}$

$$\frac{P}{\rho} = \frac{a^2}{\gamma}$$

$$\therefore h = \frac{a^2}{\gamma - 1}.$$

Sub in Eqn ③:

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$$\boxed{\frac{a_1^2}{\gamma-1}+\frac{V_1^2}{2}=\frac{a_2^2}{\gamma-1}+\frac{V_2^2}{2}} \longrightarrow (A)$$

when $M=1 \Rightarrow V=a$

$$\left[\frac{V}{a}=1\right]$$

critical $\rightarrow M=M^*$
(∴ choked flow) So $V=a^*$

Sub in Eqn (A)

$$\frac{a^{*2}}{\gamma-1}+\frac{a^{*2}}{2} = \frac{2a^{*2}+a^{*2}\gamma-a^{*2}}{2(\gamma-1)}$$

$$=\frac{a^{*2}(\gamma+1)}{2(\gamma-1)} \longrightarrow (B)$$

Eqn (A) = Eqn (B)

$$\boxed{\frac{a_1^2}{\gamma-1}+\frac{V_1^2}{2}=\frac{a_2^2}{\gamma-1}+\frac{V_2^2}{2}=\frac{a^{*2}(\gamma+1)}{2(\gamma-1)}} \longrightarrow (C)$$

From the above eqn: (To find a_1 consider (1) & (3)rd terms)

$$\frac{a_1^2}{\gamma-1}+\frac{V_1^2}{2}=\frac{a^{*2}(\gamma+1)}{2(\gamma-1)}.$$

$$a_1^2=\left[\frac{a^{*2}(\gamma+1)}{2(\gamma-1)}-\frac{V_1^2}{2}\right]\gamma-1$$

$$\boxed{a_1^2=\frac{a^{*2}(\gamma+1)}{2}-\frac{V_1^2(\gamma-1)}{2}}$$

|||ly $\boxed{a_2^2=\frac{a^{*2}(\gamma+1)}{2}-\left(\frac{\gamma-1}{2}\right)V_2^2}$

Since the flow is adiabatic across the shock wave, a^* in the above relations for a_1^2 & a_2^2 has the same constant value.

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Now substitute the values of a_1^2 & a_2^2 in the eqn of (7):

$$\frac{a_1^2}{\gamma V_1} - \frac{a_2^2}{\gamma V_2} = V_2 - V_1$$

$$\frac{\frac{a^{*2}(\gamma+1)}{2} - V_1^2\left(\frac{\gamma-1}{2}\right)}{\gamma V_1} - \frac{a^{*2}\left(\frac{\gamma+1}{2}\right) - \left(\frac{\gamma-1}{2}\right)V_2^2}{\gamma V_2} = V_2 - V_1$$

$$\frac{\gamma+1}{2\gamma} a^{*2}\left[\frac{1}{V_1}-\frac{1}{V_2}\right] + \frac{\gamma-1}{2\gamma}\left[\frac{V_2^2}{V_2}-\frac{V_1^2}{V_1}\right] = V_2 - V_1.$$

$$\frac{\gamma+1}{2\gamma} a^{*2}\left[\frac{V_2-V_1}{V_1V_2}\right] + \frac{\gamma-1}{2\gamma}\left[V_2-V_1\right] = V_2-V_1.$$

$$\frac{\gamma+1}{2\gamma V_1V_2}a^{*2} + \frac{\gamma-1}{2\gamma} = 1.$$

$$a^{*2} = \left(1-\frac{\gamma-1}{2\gamma}\right)\frac{2\gamma V_1V_2}{\gamma+1}$$

$$a^{*2} = \frac{2\gamma V_1V_2}{\gamma+1} - \frac{(\gamma-1)2\gamma V_1V_2}{2\gamma(\gamma+1)}$$

$$a^{*2} = \left(\frac{2\gamma-\gamma+1}{2\gamma}\right)\frac{2\gamma V_1V_2}{\gamma+1}$$

or $= V_1V_2\left(\frac{2\gamma}{\gamma+1} - \frac{(\gamma-1)}{\gamma+1}\right) = V_1V_2\left(\frac{2\gamma-\gamma+1}{\gamma+1}\right) = V_1V_2$

$$a^{*2} = \frac{\gamma+1}{\gamma+1}V_1V_2 \Rightarrow \boxed{a^{*2}=V_1V_2} \rightarrow (8)$$

This is called Prandtl relation.

we know, $M^* = \frac{V}{a^*}$ in terms of speed ratio

$\therefore M_2^* = \frac{V_2}{a^*}$ & $M_1^* = \frac{V_1}{a^*}$

$\therefore a^* = \frac{V_2}{M_2^*}$ & $a^* = \frac{V_1}{M_1^*}$

Multiply both eqns:

$$a^*\times a^* = \frac{V_2}{M_2^*}\times\frac{V_1}{M_1^*}$$

$$a^{*2} = \frac{V_1 V_2}{M_2^* M_1^*}$$

Sub in Eqn (5):

$$a^{*2} = V_1 V_2 .$$

$$\frac{V_1 V_2}{M_2^* M_1^*} = V_1 V_2 \Longrightarrow \frac{1}{M_2^* M_1^*} = 1 \Longrightarrow M_2^* M_1^* = 1.$$

$$\boxed{M_2^* = \frac{1}{M_1^*}} \rightarrow (9)$$

This eqn implies that the velocity change across a normal shock must be from supersonic to subsonic & vice versa. But practically change from subsonic to supersonic is not possible.

So that we can tell that mach number behind a normal shock is always subsonic.

Consider Eqn (c):

$$\frac{a_1^2}{\gamma-1} + \frac{V_1^2}{2} = \frac{a_2^2}{\gamma-1} + \frac{V_2^2}{2} = \frac{a^{*2}(\gamma+1)}{2(\gamma-1)}$$

$$\frac{a^2}{\gamma-1} + \frac{V^2}{2} = \frac{a^{*2}(\gamma+1)}{2(\gamma-1)}.$$

$\div$ throughout by V^2.

$$\frac{\left(\frac{a}{V}\right)^2}{\gamma-1} + \frac{1}{2} = \frac{\gamma+1}{2(\gamma-1)}\left(\frac{a^*}{V}\right)^2$$

$$\left[\because \frac{V}{a} = M \right.$$
$$\left. \& \ \frac{V}{a^*} = M^* \right]$$

$$\frac{(1/M)^2}{\gamma-1} + \frac{1}{2} = \frac{\gamma+1}{2(\gamma-1)}\left(\frac{1}{M^*}\right)^2$$

$$\frac{\gamma+1}{2(\gamma-1)} \times \frac{1}{M^{*2}} = \frac{1}{M^2(\gamma-1)} + \frac{1}{2}.$$

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$$\cancel{\frac{\gamma+1}{2(\gamma-1)} - \frac{1}{M^2(\gamma-1)} - \frac{1}{2} = M^{*2}}$$

$$\frac{1}{M^{*2}} = \left[\frac{1}{M^2(\gamma-1)} + \frac{1}{2}\right]\frac{2(\gamma-1)}{\gamma+1}$$

$$\frac{1}{M^{*2}} = \frac{2}{M^2(\gamma+1)} + \frac{\gamma-1}{\gamma+1} = \frac{2(\gamma+1) + M^2(\gamma-1)(\gamma+1)}{M^2(\gamma+1)^2}$$

$$\boxed{M^{*2} = \frac{M^2(\gamma+1)}{2+M^2(\gamma-1)}} \longrightarrow (10)$$

Replace M_1^* and M_2^* in eqn (9) by eqn (10), we get:

$$\frac{M_2^2(\gamma+1)}{2+M_2^2(\gamma-1)} = \frac{1}{\dfrac{M_1^2(\gamma+1)}{2+M_1^2(\gamma-1)}}$$

$$\frac{M_2^2(\gamma+1)}{2+M_2^2(\gamma-1)} = \frac{2+M_1^2(\gamma-1)}{M_1^2(\gamma+1)}$$

$$\frac{M_2^2(\gamma+1)}{2} + \frac{\gamma+1}{\gamma-1} = \frac{2}{M_1^2(\gamma+1)} + \frac{\gamma-1}{\gamma+1}$$

$$M_2^2 = \left[\frac{2}{M_1^2(\gamma+1)} + \frac{\gamma-1}{\gamma+1} - \frac{\gamma+1}{\gamma-1}\right]\frac{2}{\gamma+1}$$

$\longrightarrow$ Derive (H.W).

$$\boxed{M_2^2 = \frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2 - \left(\frac{\gamma-1}{2}\right)}} \longrightarrow (11)$$

Eq (11) shows that for a perfect gas the mach number

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H.W

(1) $\Rightarrow$ $M^* = \dfrac{M^2(r+1)}{2+M^2(r-1)}$ (Derive M_2^2 from M^*).

$$M_1^* = \frac{M_1^2(r+1)}{2+M_1^2(r-1)} \quad \text{and} \quad M_2^* = \frac{M_2^2(r+1)}{2+M_2^2(r-1)}$$

$$M_2^* = \frac{1}{M_1^*} \Rightarrow \frac{M_2^2(r+1)}{2+M_2^2(r-1)} = \frac{1}{\dfrac{M_1^2(r+1)}{2+M_1^2(r-1)}}$$

$$\frac{M_2^2 r + M_2^2}{2+M_2^2 r - M_2^2} = \frac{2+M_1^2 r - M_1^2}{M_1^2 r + M_1^2}$$

$$\frac{\cancel{M_2^2}(r+1)}{\cancel{M_2^2}\left(\frac{2}{M_2^2}+r-1\right)} = \frac{\cancel{M_1^2}\left(\frac{2}{M_1^2}+r-1\right)}{\cancel{M_1^2}(r+1)}$$

$$(r+1)^2 = \left(\frac{2}{M_1^2}+(r-1)\right)\left(\frac{2}{M_2^2}+(r-1)\right)$$

$$(r+1)^2 = \frac{4}{M_1^2 M_2^2} + \frac{2(r-1)}{M_1^2} + \frac{2(r-1)}{M_2^2} + (r-1)^2$$

$$(r+1)^2 = \frac{4}{M_1^2 M_2^2} + 2(r-1)\left[\frac{1}{M_1^2}+\frac{1}{M_2^2}\right] + (r-1)^2$$

$$(r+1)^2 = \frac{4}{M_1^2 M_2^2} + 2(r-1)\left[\frac{M_2^2+M_1^2}{M_1^2 M_2^2}\right] + (r-1)^2$$

$$(r+1)^2 = \frac{4}{M_1^2 M_2^2} + \frac{2rM_2^2 + 2rM_1^2 - 2M_2^2 - 2M_1^2}{M_1^2 M_2^2} + (r-1)^2$$

$$(r+1)^2 - (r-1)^2 = \frac{4 + 2rM_2^2 + 2rM_1^2 - 2M_2^2 - 2M_1^2}{M_1^2 M_2^2}$$

$$\cancel{r^2} + 2r + \cancel{1} - \cancel{r^2} + 2r - \cancel{1} = \frac{4 + 2rM_2^2 + 2rM_1^2 - 2M_2^2 - 2M_1^2}{M_1^2 M_2^2}$$

$$4rM_1^2 M_2^2 = 4 + 2rM_2^2 + 2rM_1^2 - 2M_2^2 - 2M_1^2$$

$$2rM_1^2 M_2^2 = 2 + rM_2^2 + rM_1^2 - M_2^2 - M_1^2.$$

$$2rM_1^2 M_2^2 - rM_2^2 - rM_1^2 = 2 - M_2^2 - M_1^2$$

$$-r\left[M_1^2 + M_2^2 - 2M_1^2 M_2^2\right] = 2 - M_2^2 - M_1^2$$

$$-rM_1^2 + M_1^2 + 2rM_1^2M_2^2 = rM_2^2 - M_2^2 + 2$$

$$M_1^2(1-r) = rM_2^2 - M_2^2 - 2rM_1^2M_2^2 + 2.$$

$$M_1^2(1-r) = M_2^2(r-1-2rM_1^2) + 2.$$

$$M_2^2 = \frac{M_1^2(1-r)-2}{r-1-2rM_1^2}$$

$$\therefore M_2^2 = \frac{1 + \frac{M_1^2(r-1)}{2}}{rM_1^2 - \frac{(r-1)}{2}}$$ (By taking -2 outside).

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behind the normal shock is a function of only the mach number ahead of the shock.

Cases:

(i) when M_1 increases above 1. ($M_1 > 1$), M_2 decreases. M_2 becomes progressively less than 1. so we can tell that normal shock became more stronger.

(ii) When M_1 tends to infinity ($M_1 \to \infty$), M_2 approaches a finite minimum value. $M_2 = \sqrt{\frac{\gamma-1}{2\gamma}}$

For air $\gamma = 1.4$ (at standard condition).

$M_2 = 0.378$

Ratio of velocities $\left(\frac{V_1}{V_2}\right)$ can be written as:

$$\boxed{\frac{V_1}{V_2} = \frac{V_1^2}{V_1 V_2} = \frac{V_1^2}{a^{*2}} = M_1^{*2}} \longrightarrow (12) \quad (\times \, \& \div \text{ by } V_1)$$

Using eq (10) & (12); we can derive other normal shock relationships.

Consider eq (1): $\rho_1 V_1 = \rho_2 V_2$.

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2}.$$

From eq (12): $$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = M_1^{*2}$$

from eqn (10): $$M_1^{*2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}$$

i.e., $$\boxed{\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}} \longrightarrow (13)$$

from momentum eqn (2):

$$P_1 + \rho_1 V_1^2 = P_2 + \rho_2 V_2^2$$

$$P_2 - P_1 = \rho_1 V_1^2 - \rho_2 V_2^2$$

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$$= \rho_1 V_1^2 - (\rho_1 V_1) V_2 \qquad \left\{ \because \rho_1 V_1 = \rho_2 V_2 \atop \text{from continuity equation} \right\}$$

$$= \rho_1 V_1 \left[V_1 - V_2 \right]$$

$$\therefore P_2 - P_1 = \rho_1 V_1^2 \left[1 - \frac{V_2}{V_1} \right]$$

Divide throughout by P_1.

$$\frac{P_2 - P_1}{P_1} = \frac{\rho_1 V_1^2}{P_1} \left[1 - \frac{V_2}{V_1} \right] \longrightarrow (14)$$

Recalling $a_1^2 = \frac{\gamma P_1}{\rho_1} \Rightarrow \frac{a_1^2}{\gamma} = \frac{P_1}{\rho_1}$

$$\therefore \frac{P_2 - P_1}{P_1} = \frac{\gamma V_1^2}{a_1^2} \left[1 - \frac{V_2}{V_1} \right]$$

$$\therefore \frac{P_2 - P_1}{P_1} = \gamma M_1^2 \left[1 - \frac{V_2}{V_1} \right] \rightarrow (15) \qquad \left[\because M^2 = \frac{V^2}{a^2} \right]$$

Substitute the value of $\frac{V_2}{V_1}$ from eqn (13).

$$\frac{P_2 - P_1}{P_1} = \gamma M_1^2 \left[1 - \frac{2 + (\gamma - 1) M_1^2}{(\gamma + 1) M_1^2} \right]$$

$$\frac{P_2}{P_1} = 1 + \gamma M_1^2 \left[1 - \frac{(2 + (\gamma - 1) M_1^2)}{(\gamma + 1) M_1^2} \right]$$

$$\frac{P_2}{P_1} = 1 + \gamma M_1^2 \left[\frac{(\gamma + 1) M_1^2 - 2 - (\gamma - 1) M_1^2}{(\gamma + 1) M_1^2} \right]$$

$$\therefore \frac{P_2}{P_1} = 1 + \gamma M_1^2 \left[\frac{\cancel{\gamma M_1^2} + M_1^2 - 2 - \cancel{\gamma M_1^2} + M_1^2}{(\gamma + 1) M_1^2} \right]$$

$$= 1 + \gamma \cancel{M_1^2} \left[\frac{2 M_1^2 - 2}{(\gamma + 1) \cancel{M_1^2}} \right]$$

$$\therefore \boxed{\frac{P_2}{P_1} = 1 + \frac{2\gamma}{(\gamma + 1)} \left[M_1^2 - 1 \right]} \longrightarrow (16)$$

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The ratio $\frac{P_2 - P_1}{P_1} = \frac{\Delta P}{P_1}$ is called the shock strength

The state eqn $P = \rho R T$ can be used to get the ratio

$$P_1 = \rho_1 R T_1 \; ; \; P_2 = \rho_2 R T_2$$

$$\frac{T_2}{T_1} = \frac{P_2}{P_1} \times \frac{\rho_1}{\rho_2} \longrightarrow (17)$$

Substitute eqn (13) & (16):

$$\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)\right]\left[\frac{2 + (\gamma-1)M_1^2}{(\gamma+1)M_1^2}\right]$$

$$\frac{T_2}{T_1} = 1 + \frac{2(\gamma-1)(\gamma M_1^2 + 1)(M_1^2 - 1)}{(\gamma+1)^2 M_1^2}$$

$$\therefore \boxed{\frac{T_2}{T_1} = \frac{h_2}{h_1} = \frac{a_2^2}{a_1^2} = 1 + \frac{2(\gamma-1)(\gamma M_1^2 + 1)(M_1^2 - 1)}{(\gamma+1)^2 M_1^2}} \rightarrow (18)$$

we know

$$s_2 - s_1 = C_p \ln\left(\frac{T_2}{T_1}\right) - R \ln \frac{P_2}{P_1}$$

Substitute the values of $\frac{P_2}{P_1}$ & $\frac{T_2}{T_1}$ in above equation of $s_2 - s_1$

$$\boxed{s_2 - s_1 = C_p \ln\left[1 + \frac{2(\gamma-1)(\gamma M_1^2 + 1)(M_1^2 - 1)}{(\gamma+1)^2 M_1^2}\right] - R \ln\left[1 + \frac{2\gamma}{(\gamma+1)}(M_1^2 - 1)\right]} \rightarrow (19)$$

$$\frac{h_2}{h_1} = \frac{C_p T_2}{C_p T_1}$$

$$\therefore \boxed{\frac{h_2}{h_1} = \frac{T_2}{T_1}}$$

$$\frac{a_2}{a_1} = \frac{\sqrt{\gamma R T_2}}{\sqrt{\gamma R T_1}}$$

$$\boxed{\frac{a_2^2}{a_1^2} = \frac{T_2}{T_1}}$$

⇒ All the above equations (M_2, $s_2 - s_1$, $\frac{P_2}{P_1}$, $\frac{T_2}{T_1}$, $\frac{\rho_2}{\rho_1}$) are all functions of Mach number only (M_1)

⇒ This explains the importance of mach number in the quantitative governance of compressible flow.

⇒ These results are reasonably accurate upto $M_1 = 5$ for at standard conditions.

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⇒ Beyond $M_{a1} = 5$, temperature behind normal shock will be very high. Then 'γ' will not be constant and we cannot apply the above relations.

$$\left(1 + \frac{2\gamma M_1^2}{\gamma+1} - \frac{2\gamma}{\gamma+1}\right)\left(\frac{2}{(\gamma+1)M_1^2} + \frac{(\gamma-1)}{\gamma+1}\right)$$

$$\frac{2}{(\gamma+1)M_1^2} + \frac{\gamma-1}{\gamma+1} + \frac{4\gamma}{(\gamma+1)^2} + \frac{2\gamma M_1^2(\gamma-1)}{(\gamma+1)^2} - \frac{4\gamma}{(\gamma+1)^2 M_1^2} - \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}$$

$$\frac{2}{(\gamma+1)M_1^2} + \frac{\gamma-1}{\gamma+1} + \frac{4\gamma + 2\gamma M_1^2(\gamma-1) - 2\gamma(\gamma-1)}{(\gamma+1)^2} - \frac{4\gamma}{(\gamma+1)^2 M_1^2}$$

$$\frac{2(\gamma+1) + (\gamma-1)(\gamma+1)M_1^2}{(\gamma+1)^2 M_1^2} + \frac{4\gamma + 2\gamma^2 M_1^2 - 2\gamma M_1^2 - 2\gamma^2 + 2\gamma}{(\gamma+1)^2} - \frac{4\gamma}{(\gamma+1)^2 M_1^2}$$

$$= \frac{2\gamma + 2 + (\gamma^2-1)M_1^2 - 4\gamma}{(\gamma+1)^2 M_1^2} + \frac{4\gamma + 2\gamma^2 M_1^2 - 2\gamma M_1^2 - 2\gamma^2 + 2}{(\gamma+1)^2}$$

⇒ When $M_1 \to \infty$

$$\lim_{M_1 \to \infty} M_2 = \sqrt{\frac{\gamma-1}{2\gamma}} \quad \checkmark \qquad \frac{-2\gamma + 2 + (\gamma^2-1)}{(\gamma+1)^2 M_1^2} +$$

$$\lim_{M_1 \to \infty} \frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1} \quad \checkmark$$

$$\lim_{M_1 \to \infty} \frac{P_2}{P_1} = \infty \quad \checkmark$$

$$\lim_{M_1 \to \infty} \frac{T_2}{T_1} = \infty \quad \checkmark$$

⇒ When $M_1 = 1$

$$M_2 = \frac{\rho_2}{\rho_1} = \frac{P_2}{P_1} = \frac{T_2}{T_1} = 1.$$

i.e., No finite changes across the wave.

* From Prandtl eqn: It is possible to decelerate the flow from supersonic to subsonic & vice-versa across a normal shock wave, only the former is physically feasible. Justify the statement:

⟶ When $M_1 = 1$, $\Delta S = 0$.
When $M_1 < 1$, $\Delta S =$ less than 0 (−ve)
When $M_1 > 1$, $\Delta S =$ greater than 0 (+ve)
} For eqn (19).

⟹ From IInd law of thermodynamics $\Delta S \geq 0$. So M_1

must be greater than or equal to 1.

$\Rightarrow$ when $M_1 < 1$; the entropy across the wave $\downarrow$ ses, which is impossible.

$\therefore$ Subsonic to supersonic conversion is not possible.

$\therefore M_1 > 1$ across a normal shock.

when $M_1 > 1$, M_2 will be less than 1 (from eq (11))

when $M_1 > 1$, $\frac{\rho_2}{\rho_1} > 1$ and $\frac{P_2}{P_1} > 1$ [from (13) & (16)] (since here only static pressure is considered. $\therefore P_2/P_1 > 1$)

$M_1 > 1 \Rightarrow \frac{T_2}{T_1} > 1$. (from eq (18)).

The changes in properties takes place at a distance of 10^{-5} cm in the normal shock. The velocity as well as temperature gradient will be larger. It will result in the increase in entropy across the shock.

* Hugoniot Equation: The static pressure always $\uparrow$ ses across a shock wave. from the continuity equation, we know that:

$$\rho_1 V_1 = \rho_2 V_2 \longrightarrow (1)$$

$$\therefore \boxed{V_2 = V_1 \cdot \frac{\rho_1}{\rho_2}} \longrightarrow (2)$$

Consider the momentum equation:

$$P_1 + \rho_1 V_1^2 = P_2 + \rho_2 V_2^2 \longrightarrow (3)$$

Substitute eq (2) in eq (3):

$$P_1 + \rho_1 V_1^2 = P_2 + \rho_2 \left(V_1 \frac{\rho_1}{\rho_2}\right)^2$$

$$\rho_1 V_1^2 - \rho_2 \left(V_1 \frac{\rho_1}{\rho_2}\right)^2 = P_2 - P_1$$

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$$V_1^2\left[\rho_1 - \rho_2\frac{\rho_1^2}{\rho_2^2}\right] = P_2 - P_1$$

$$\therefore V_1^2 = \frac{P_2-P_1}{\rho_1 - \frac{\rho_1^2}{\rho_2}} = \frac{P_2-P_1}{\frac{\rho_1\rho_2-\rho_1^2}{\rho_2}} = \frac{(P_2-P_1)\rho_2}{\rho_1(\rho_2-\rho_1)}$$

$$\boxed{V_1^2 = \frac{P_2-P_1}{\rho_2-\rho_1}\left(\frac{\rho_2}{\rho_1}\right)} \longrightarrow (4)$$

Substitute the value of V_1^2 in Eqn (2):

$$V_2^2 = V_1^2\frac{\rho_1^2}{\rho_2^2}$$

$$V_2^2 = \frac{P_2-P_1}{\rho_2-\rho_1}\left(\frac{\rho_2}{\rho_1}\right)\left(\frac{\rho_1^2}{\rho_2^2}\right)$$

$$\boxed{V_2^2 = \frac{P_2-P_1}{\rho_2-\rho_1}\frac{\rho_1}{\rho_2}} \longrightarrow (5)$$

Now consider the energy equation:

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$$

we know that: $h = e + \frac{P}{\rho}$ (from $H = E + PV$)

Substitute the value of h in energy eqn:

$$\boxed{e_1 + \frac{P_1}{\rho_1} + \frac{V_1^2}{2} = e_2 + \frac{P_2}{\rho_2} + \frac{V_2^2}{2}} \longrightarrow (6)$$

we can substitute the values of V_1^2 & V_2^2 in eqn (6):

$$e_1 + \frac{P_1}{\rho_1} + \frac{\frac{P_2-P_1}{\rho_2-\rho_1}\left(\frac{\rho_2}{\rho_1}\right)}{2} = e_2 + \frac{P_2}{\rho_2} + \frac{\frac{P_2-P_1}{\rho_2-\rho_1}\left(\frac{\rho_1}{\rho_2}\right)}{2}$$

$$e_2 - e_1 = \frac{P_1}{\rho_1} - \frac{P_2}{\rho_2} + \frac{1}{2}\left(\frac{P_2-P_1}{\rho_2-\rho_1}\right)\left(\frac{\rho_2}{\rho_1}\right) - \frac{1}{2}\left(\frac{P_2-P_1}{\rho_2-\rho_1}\right)\left(\frac{\rho_1}{\rho_2}\right)$$

$$= \frac{P_1\rho_2 - P_2\rho_1}{\rho_1\rho_2} + \frac{1}{2}\frac{P_2-P_1}{\rho_2-\rho_1}\left[\frac{\rho_2}{\rho_1} - \frac{\rho_1}{\rho_2}\right]$$

$$= \frac{P_1\rho_2 - P_2\rho_1}{\rho_1\rho_2} + \frac{1}{2}\frac{P_2-P_1}{\rho_2-\rho_1}\left[\frac{\rho_2^2-\rho_1^2}{\rho_1\rho_2}\right]$$

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$$= \frac{P_1\rho_2 - P_2\rho_1}{\rho_1\rho_2} + \frac{1}{2}\left[\frac{P_2-P_1}{\cancel{\rho_2-\rho_1}}\right]\left[\frac{\cancel{(\rho_2-\rho_1)}(\rho_2+\rho_1)}{\rho_1\rho_2}\right]$$

$$= \frac{P_1\rho_2 - P_2\rho_1}{\rho_1\rho_2} + \frac{1}{2}\left[\frac{P_2\rho_2 + P_2\rho_1 - P_1\rho_2 - P_1\rho_1}{\rho_1\rho_2}\right]$$

$$= P_1\left[\frac{1}{2\rho_1} - \frac{1}{2\rho_2}\right] + P_2\left[\frac{1}{2\rho_1} - \frac{1}{2\rho_2}\right].$$

$$\Rightarrow e_2 - e_1 = \frac{1}{2}\left[\frac{1}{\rho_1} - \frac{1}{\rho_2}\right]\left[P_1 + P_2\right]$$

$$\therefore \boxed{e_2 - e_1 = \frac{P_1+P_2}{2}\left[\frac{1}{\rho_1} - \frac{1}{\rho_2}\right]} \longrightarrow (7)$$

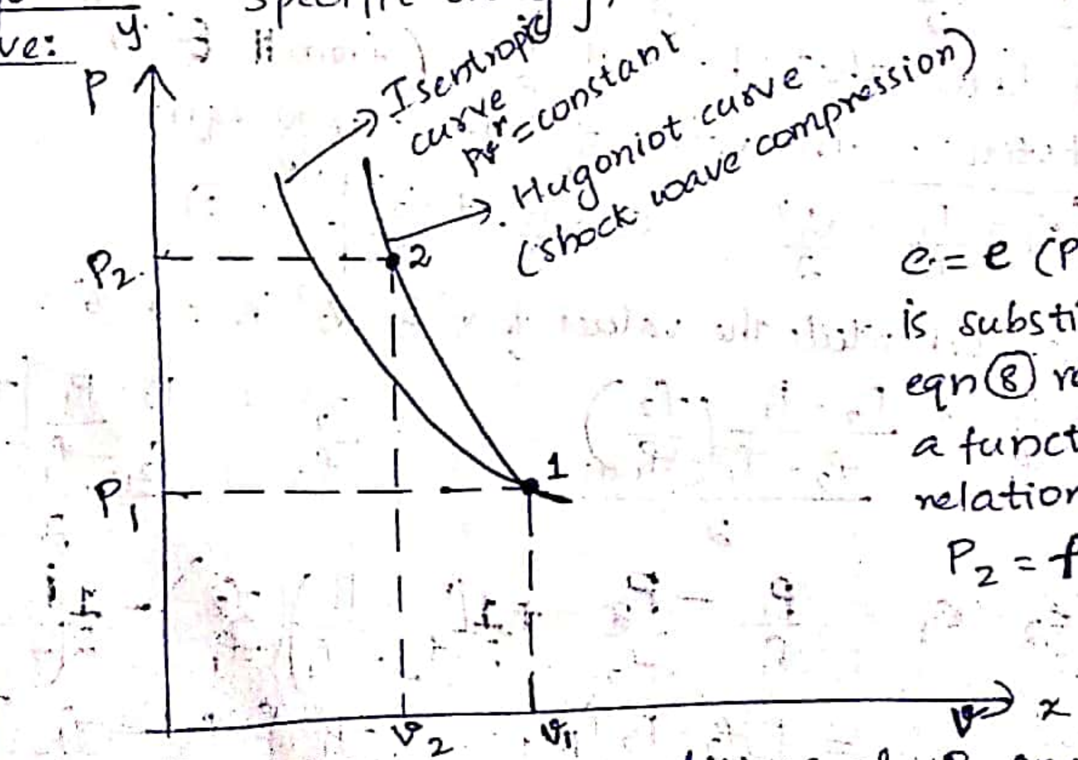
$\frac{P_1}{\rho_1} - \frac{P_2}{\rho_2} + \frac{1}{2}\left[\frac{P_2}{\rho_1}\right.$ $+ \frac{P_2}{\rho_2} - \frac{P_1}{\rho_2} - \frac{P_1}{\rho_1}$ $P_1\left[\frac{1}{\rho_1} - \frac{1}{2\rho_1} - \frac{1}{2\rho_2}\right]$ $P_2\left[-\frac{1}{\rho_2} + \frac{1}{2\rho_1} + \frac{1}{2\rho_2}\right]$

Replacing ρ_1 & ρ_2 by specific volumes v_1 & v_2:
$\left(v_1 = \frac{1}{\rho_1} \text{ \& } v_2 = \frac{1}{\rho_2}\right)$.

$$\boxed{e_2 - e_1 = \frac{P_1+P_2}{2}\left[v_1 - v_2\right]} \longrightarrow (8) \text{ OR } \Delta e = P_{avg}(\Delta v_{specific})$$

This is called Hugoniot Equation

* Hugoniot curve: Specific energy, $e = e(Pv)$



$e = e(Pv)$ relation is substituted in eqn (8) resulting in a functional relation
$P_2 = f(P_1, v_1, v_2)$ → (9)

For given conditions of P_1 and v_1 upstream of the normal shock, eqn (9) represents P_2 as a function v_2. A plot of this relation on a P-v

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graph is called the Hugoniot curve. This curve is the locus of all possible pressure-volume conditions behind normal shocks of various strengths for one specific set of upstream values for P_1 and v_1 (point 1). Each point on the hugoniot curve therefore represents a different shock with a different upstream velocity u_1.

• Hugoniot equation relates only thermodynamic quantities across a shock wave. There is no assumption made for calorifically perfect gas to derive this equation, so that Hugoniot equation is a general equation holds for perfect gas, chemically reactive gas, real gas etc...

$$\Delta e = P_{avg} (\Delta V_{specific}).$$

This eqn is similar to that of equation for 1st law of thermodynamics for adiabatic process. Here internal energy 'e' is a function of pressure and specific volume.

* Normal shock wave relations (Continuation):

$\frac{T_2}{T_1}, \frac{P_2}{P_1}, \frac{\rho_2}{\rho_1}, S_2-S_1$ and M_2 are all functions of mach number only.

⇒ These gradients internal to the shock provides heat conduction and viscous dissipation that render this shock process internally irreversible.

⇒ Total Pressure and temperature effect:

→ Adiabatic process is

$$\boxed{T_{02} = T_{01}} \qquad \left[\begin{array}{l} h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \\ C_p T_{01} = C_p T_{02} \end{array}\right]$$

Total temperature is constant across a stationary normal shock wave.

→ Consider the equation of entropy for adiabatic process:

$$S_{02} - S_{01} = C_p \ln\left(\frac{T_{02}}{T_{01}}\right)^{\to 0} - R \ln\left(\frac{P_{02}}{P_{01}}\right) \qquad [\because T_{02} = T_{01}]$$

$$S_{02} - S_{01} = -R \ln\left(\frac{P_{02}}{P_{01}}\right)$$

$$S_{02} - S_{01} = R \ln\left(\frac{P_{01}}{P_{02}}\right)$$

→ Entropy varies only when there are losses in pressure. It is independent of velocity and hence there is nothing like stagnation entropy. Therefore:

$$S_2 - S_1 = R \ln\left(\frac{P_{01}}{P_{02}}\right)$$

$$\therefore \boxed{\frac{P_{02}}{P_{01}} = e^{-(S_2 - S_1)/R}}$$ (By taking exponential).

we know that $S_2 - S_1 > 0$ for normal shock wave. So $P_{02} < P_{01}$. The total pressure ↓ses across a normal shock wave.

* Rayleigh supersonic pitot tube:

In gas dynamics there are 3 types of pressure:

(i) Static pressure (P)

(ii) q, Dynamic pressure ($\frac{1}{2}\rho V^2$)

(iii) Total pressure — static + dynamic pressure or stagnation pressure (P_0)

(iv) Geometric pressure: Pressure due to the Gravitational action on the static fluid.

Total pressure is pressure corresponding to the fluid when resting isentropically. The fluid velocity will be zero. Therefore it corresponds to undistributed static pressure will be equal in all direction.

In space static pressure and the undistributed

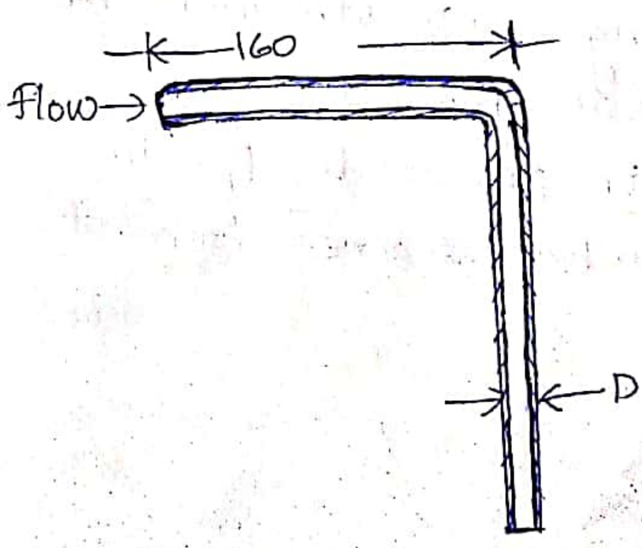
* Difference b/w the stagnation pressure and the undistributed static pressure.

⇒ for incompressible flow, these 3 pressures are linked together by Bernoulli's equation:

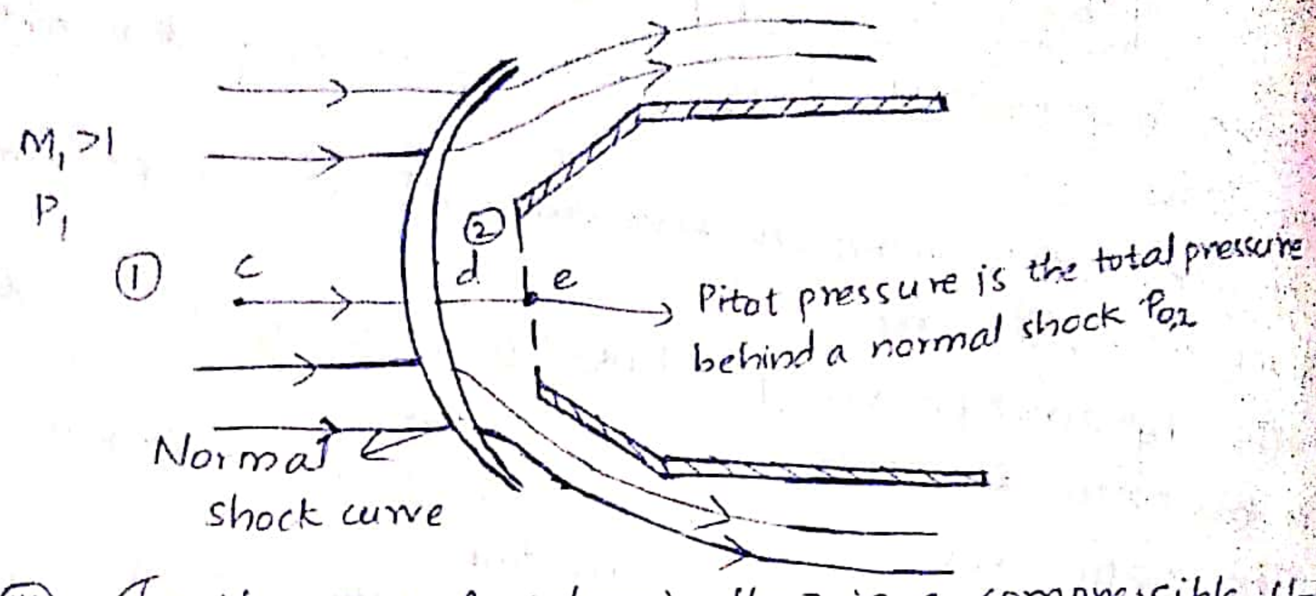
$$\boxed{P_0 - P = \frac{1}{2}\rho v^2 = q} \Rightarrow \text{(Incompressible)}$$

- In an incompressible flow P_0 will be same everywhere with no losses.
- for constant P_0 when the flow velocity ↑ses static pressure ↓ses and vice-versa.
- The dynamic pressure 'q' is linked to the K.E of the flow & it has same direction as that of the flow.
- The static pressure of a flow is that pressure which is acting normal to the flow direction.
- Therefore the total pressure also has a definite direction.
- We use pitot probe to measure this total pressure.
- Pitot tube is a tube with blunt end facing the flow.
- At blunt end pressure hole will be formed. we can measure the total pressure from here. Inside and outside diameter are considered from $\frac{1}{2}$ to 3. Length is from 15 to 20.
- In the case of supersonic flow a shock will be a detached and standing infront of the blunt end. Therefore it measures the total pressure behind the normal shock.

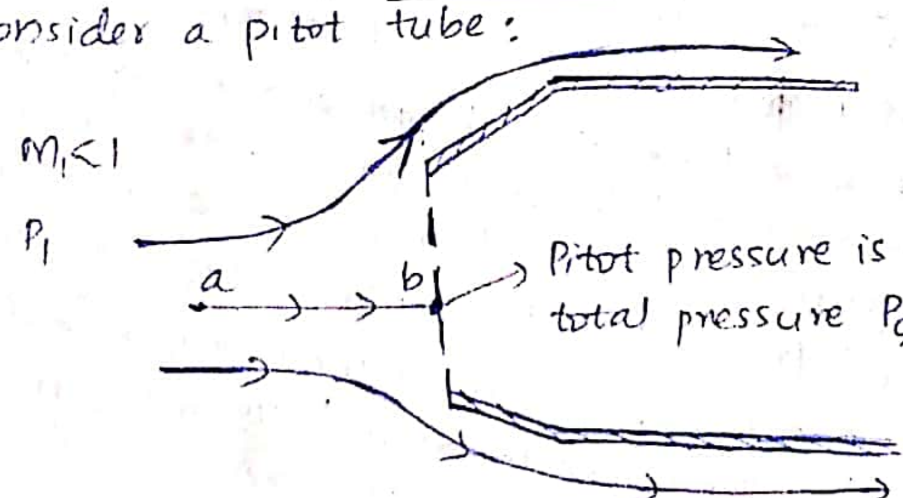
* Pitot tube:



(*) Supersonic flow over a blunt body: (fig a)



(*) • In the case of subsonic flow in a compressible flow consider a pitot tube:



ii • At point 'b' the pressure will be sensed, it is the freestream total pressure (P_{01}). As Mouth of tube, Point 'b' is the stagnation region.

iii • P_{01} is the pitot pressure read at the end of the tube.

i • The fluid element moving along streamline ab is brought to rest isentropically at point 'b'.

iv • Let the free stream static pressure be P_1. The mach number in region 1 can be attained from the equation:

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$$\boxed{\frac{P_{01}}{P_1} = \left[1 + \frac{\gamma-1}{2} M_1^2\right]^{\frac{\gamma}{\gamma-1}}} \rightarrow (1)$$

$$\therefore \boxed{M_1^2 = \frac{2}{\gamma-1}\left[\left(\frac{P_{0,1}}{P_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]} \rightarrow (2)$$

; where $P_{0,1} \rightarrow$ Pitot pressure.

$M_1 = \frac{u_1}{a_1}$, $M_1^2 = \frac{u_1^2}{a_1^2}$

$$u_1^2 = M_1^2 a_1^2$$

$$\therefore \boxed{u_1^2 = \frac{2}{\gamma-1}\left[\left(\frac{P_{0,1}}{P_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] a_1^2} \rightarrow (3)$$

(*) • In the case of <u>supersonic flow</u> we are considering the flow over a blunt body given by fig (a). free stream is supersonic & pitot tube presents an obstruction to the flow, a strong bow shock wave will form in front of the tube.

• e is the stagnation region.

• Fluid element is moving along the streamline cde and at e it will be at rest.

• First at point d, fluid element will be decelerated non isentropically to a subsonic velocity due to the obstruction.

• At point e, it is again isentropically decelerated to velocity zero because of the formation of the shock infront of the body.

• In this case, at point e we can only measure total pressure only behind the normal shock & it is denoted by $P_{0,2}$.

• Here entropy is increasing across the shock, there is loss in total pressure across the shock.

So $P_{02} < P_{01}$

$$\boxed{\frac{P_{02}}{P_1} = \frac{P_{02}}{P_2}\frac{P_2}{P_1}} \rightarrow (4) \quad (\times \,\&\, \div \text{ by } P_2).$$

where, $P_1 \Rightarrow$ freestream static pressure.

$\frac{P_{02}}{P_2} \rightarrow$ ratio of the total pressure to static pressure at region 2, immediately behind the normal shock.

$\frac{P_2}{P_1} \rightarrow$ static pressure ratios across the shock.

We know,

$$\boxed{\frac{P_{02}}{P_2} = \left[1+\frac{\gamma-1}{2}M_2^2\right]^{\frac{\gamma}{\gamma-1}}} \longrightarrow (5)$$

From normal shock relations:

$$\boxed{M_2^2 = \frac{\left[1+\frac{\gamma-1}{2}M_1^2\right]}{\gamma M_1^2-\left(\frac{\gamma-1}{2}\right)}} \longrightarrow (6).$$

Sub (6) in (5):

$$\boxed{\frac{P_{02}}{P_2} = \left[1+\frac{\gamma-1}{2}\left[\frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2-\left(\frac{\gamma-1}{2}\right)}\right]\right]^{\frac{\gamma}{\gamma-1}}} \longrightarrow (7)$$

We know from normal shock relation.

$$\boxed{\frac{P_2}{P_1} = 1+\frac{2\gamma}{\gamma+1}\left[M_1^2-1\right]} \longrightarrow (8)$$

Substitute Eqn (7) & (8) in eqn (4).

$$\frac{P_{02}}{P_1} = \left[1+\frac{\gamma-1}{2}\left[\frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2-\left(\frac{\gamma-1}{2}\right)}\right]\right]^{\frac{\gamma}{\gamma-1}}\left[1+\frac{2\gamma}{\gamma+1}\left[M_1^2-1\right]\right]$$

$$\boxed{\frac{P_{02}}{P_1} = \left[\frac{(\gamma+1)^2 M_1^2}{4\gamma M_1^2-2(\gamma-1)}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{1-\gamma+2\gamma M_1^2}{\gamma+1}\right]} \rightarrow$$

This is the Rayleigh's supersonic pitot tube formula. It

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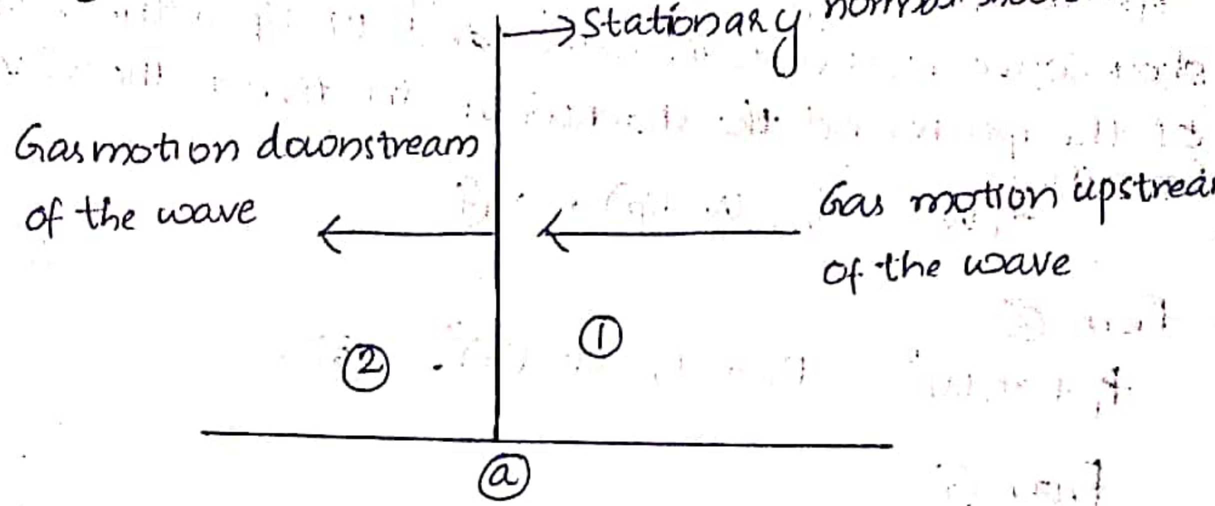
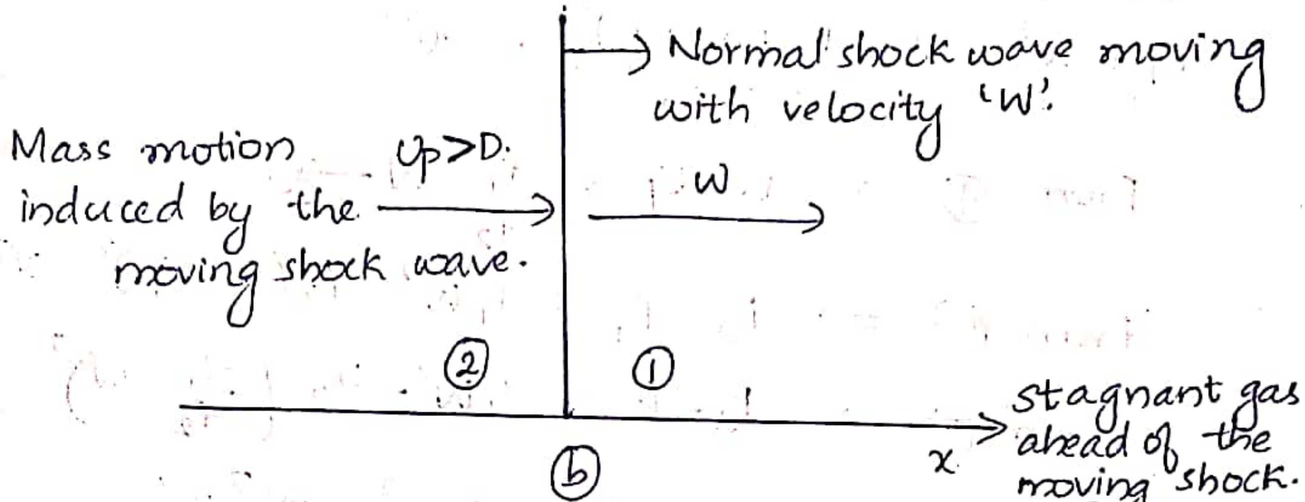
relates pitot pressure (P_{o2}), static pressure of the free stream (P_1) & free stream mach number (M_1).

* **Moving Normal Shock:**





Consider the Continuity Equation: $\rho_1 U_1 = \rho_2 U_2 \rightarrow (1)$

Consider the momentum equation: $P_1 + \rho_1 U_1^2 = P_2 + \rho_2 U_2^2 \rightarrow (2)$

Consider the energy equation: $h_1 + \frac{U_1^2}{2} = h_2 + \frac{U_2^2}{2} \rightarrow (3)$

where, U_1 = velocity of the gas ahead of the shockwave relative to the wave.

U_2 = Velocity of the gas behind the shock wave, relative to the wave.

Consider a stationary normal shock wave as shown in figure A and the above equations are corresponding to figure A.

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Now consider a moving shockwave with velocity W and a velocity induced region behind the shockwave with velocity U_p.

As shown in figure b, W = velocity of the gas ahead of the shockwave relative to the wave and $W - U_p$ is the velocity of the gas behind the shockwave relative to the wave.

From ①:

$$\rho_1 W = \rho_2 (W - U_p) \rightarrow ④$$

From ②:

$$P_1 + \rho_1 W^2 = P_2 + \rho_2 (W - U_p)^2 \rightarrow ⑤$$

From ③:

$$h_1 + \frac{W^2}{2} = h_2 + \frac{(W - U_p)^2}{2} \rightarrow ⑥$$

From ④ $\Rightarrow$ $W - U_p = \frac{\rho_1}{\rho_2} \cdot W \rightarrow ⑦$

From ⑤ $\Rightarrow$ $P_2 - P_1 = \rho_1 W^2 - \rho_2 (W - U_p)^2$

From ⑦ $\Rightarrow$ $P_2 - P_1 = \rho_1 W^2 - \rho_2 \left(\frac{\rho_1}{\rho_2} \cdot W\right)^2$

$$P_2 - P_1 = \rho_1 W^2 - \frac{\rho_1^2 W^2}{\rho_2}$$

$$\therefore P_2 - P_1 = \rho_1 W^2 \left(1 - \frac{\rho_1}{\rho_2}\right) \rightarrow ⑧$$

$$\therefore W^2 = \frac{P_2 - P_1}{\rho_2 - \rho_1} \left(\frac{\rho_2}{\rho_1}\right) \rightarrow ⑨$$

$$\frac{T_2}{T_1} = \frac{P_2}{P_1} \left(\frac{\frac{r+1}{r-1} + \frac{P_2}{P_1}}{1 + \frac{r+1}{r-1} \frac{P_2}{P_1}} \right)$$

$$\frac{\rho_2}{\rho_1} = \frac{1 + \left(\frac{r+1}{r-1}\right)\left(\frac{P_2}{P_1}\right)}{\frac{r+1}{r-1} + \frac{P_2}{P_1}}$$

we have $M_s = \frac{W}{a_1}$ where $M_s \rightarrow$ Mach number of moving Shock wave.

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}(M_s^2 - 1)$$

$$M_s = \sqrt{\frac{\gamma+1}{2\gamma}\left(\frac{P_2}{P_1} - 1\right) + 1}$$

$$W = a_1 \sqrt{\frac{\gamma+1}{2\gamma}\left(\frac{P_2}{P_1} - 1\right) + 1}$$

$$\rho_1 W = \rho_2 (W - U_p)$$

$$\therefore \rho_2 U_p = \rho_2 W - \rho_1 W$$

$$\rho_2 U_p = \rho_2 W \left[1 - \frac{\rho_1}{\rho_2}\right]$$

$$\therefore U_p = W \left[1 - \frac{\rho_1}{\rho_2}\right]$$

$$\therefore \boxed{U_p = \frac{a_1}{\gamma}\left(\frac{P_2}{P_1} - 1\right)\left[\frac{\frac{2\gamma}{\gamma+1}}{P_2/P_1 + \frac{\gamma-1}{\gamma+1}}\right]^{1/2}}$$

* Weak shock and Strong Shock:

A weakshockwave is that for which normalize the pressure jump is very small.

A strong shock is one for which $\frac{P_2}{P_1}$ is very large.